From Fourier to Gegenbauer: Dimension Walks on Spheres

Jochen Fiedler*
Institute of Applied Mathematics
University of Heidelberg

March 28, 2013

In this article we provide a solution to Problem 2 in Gneiting, 2013. Specifically, we show that the even- resp. odd-dimensional Schoenberg coefficients in Gegenbauer expansions of isotropic positive definite functions on the sphere $\mathbb{S}^d$ can be expressed as linear combinations of Fourier resp. Legendre coefficients, and we give closed form expressions for the coefficients involved in these expansions.

1 Introduction

For an integer $d \geq 1$ we denote the unit sphere in Euclidean space $\mathbb{R}^{d+1}$ equipped with the Euclidean norm by $\mathbb{S}^d := \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$. A function $h : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ is *positive definite* if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j h(x_i, x_j) \geq 0, \tag{1}$$

for all integers $n \geq 1$ and for every choice of constants $a_1, \ldots, a_n \in \mathbb{R}$ and every choice of pairwise distinct points $x_1, \ldots, x_n \in \mathbb{S}^d$. The function h is said to be *isotropic* if there exists a function $\psi : [0, \pi] \to \mathbb{R}$ such that

$$h(x, y) = \psi(\theta(x, y)), \qquad x, y \in \mathbb{S}^d,$$

where $\theta(x, y) = \arccos(\langle x, y \rangle)$ denotes the great circle distance between x and y; here $\langle ., . \rangle$ is the standard scalar product in $\mathbb{R}^{d+1}$.

We refer to $\Psi_d, d = 1, 2, \ldots,$ as the class of continuous functions $\psi : [0, \pi] \to \mathbb{R}$ with $\psi(0) = 1$ for which the associated isotropic function $h(x, y) = \psi(\theta(x, y))$ is positive definite. Therefore, Ψ_d corresponds to the class of correlation functions for mean-square continuous,

*The author's research was supported by the German Research Foundation (DFG) within the programme "Spatio/Temporal Probabilistic Graphical Models and Applications in Image Analysis", grant GRK 1653.

stationary and isotropic random fields on $\mathbb{S}^d$ (Jones, 1963). In addition, Ψ_d^+ is defined as the class of elements $\psi \in \Psi_d$ for which the associated function $h(x, y)$ is strictly positive definite, that is the sum in (1) is strictly positive.

Isotropic positive definite functions on spheres have attracted research interest in several areas. They occur as correlation functions for stationary and isotropic random fields on the sphere and, hence, are studied in the context of spatial statistics (Banerjee, 2005,Huang, Zhang, and Robeson, 2011 or Hansen, Thorarinsdottir, and Gneiting, 2011). Furthermore, they are used in approximation theory as radial basis functions for interpolating scattered data on spherical domains, see for example Xu and Cheney, 1992,Fasshauer and Schumaker, 1998 or Cavoretto and De Rossi, 2010.

Recently, Gneiting Gneiting, 2013 has reviewed conditions for functions to belong to Ψ_d or Ψ_d^+ and uses them to study parametric families of isotropic and stationary correlation functions on the sphere. In his work, he also states several problems for future research, one of which has been solved in Ziegel, 2013, and the solution to another one is given here.

The following important characterization of the members of Ψ_d and Ψ_d^+ (due to Schoenberg Schoenberg, 1942 and Chen Chen, Menegatto, and Sun, 2003) plays a key role. It says that the members of Ψ_d can be expressed as series expansions using Gegenbauer polynomials C_n^λ. In particular, the class Ψ_d, $d \geq 1$, consists of the functions of the form

$$\psi(\theta) = \sum_{n=0}^{\infty} b_{n,d} \frac{C_n^{(d-1)/2}(\cos(\theta))}{C_n^{(d-1)/2}(1)}, \qquad \theta \in [0, \pi], \tag{2}$$

with $b_{n,d} \geq 0$ and $\sum_{n=0}^{\infty} b_{n,d} = 1$. For $d \geq 2$ the class Ψ_d^+ consists of functions of Ψ_d where $b_{n,d} > 0$ for infinitely many even and infinitely many odd integers n. Consequently, it is possible to study the members of Ψ_d or Ψ_d^+ via the coefficients $b_{n,d}$ of the Gegenbauer expansion (2) which are called the *d-dimensional Schoenberg coefficients of* ψ. Since

$$\Psi_1 \supset \Psi_2 \supset \Psi_3 \supset \ldots,$$

every function in Ψ_d, $d \geq 1$, allows a Gegenbauer expansion in terms of $C_n^0(\cos\theta) = \cos(n\theta)$, $n = 0, 1, 2, \ldots$ and $b_{n,1}$, which is a Fourier cosine expansion. Similarly, if $d \geq 2$ every function $\psi \in \Psi_d$ allows an expansion in terms of Legendre polynomials, since $C_n^{1/2} = P_n$ is a Legendre polynomial. Hence, it is interesting to ask if it is possible to express higher dimensional Schoenberg coefficients in terms of Fourier or Legendre coefficients.

Referring to Wendland, 1995 we will call the expression of higher dimensional Schoenberg coefficients in terms of lower dimensional ones a *dimension walk*, because this describes a method which connects Gegenbauer expansions for different dimensions. In contrast, Wendland uses this phrase to describe the role of the operators Montée and Descente introduced by Matheron Mathéron, 1965, which work in different ways (see Daley and Porcu, 2013 for a characterization of these operators in terms of Schoenberg measures).

In general, dimension walks are interesting for studying properties of functions, because they allow transferring properties from a certain dimension to another.

The problem of expressing even and odd dimensional Schoenberg coefficients in terms of Fourier and Legendre coefficients can be answered using the following recursive identities, stated as Corollary 3 in Gneiting, 2013, which provide a connection between d-dimensional Schoenberg coefficients and lower dimensional ones. In particular, for all integers $n \geq 1$ it

is true that

$$b_{0,3} = b_{0,1} - \frac{1}{2} b_{2,1} \qquad \text{and} \qquad b_{n,3} = \frac{1}{2}(n+1)(b_{n,1} - b_{n+2,1}). \tag{3}$$

Furthermore, if $d \geq 2$, then for all integers $n \geq 0$

$$b_{n,d+2} = \frac{(n+d-1)(n+d)}{d(2n+d-1)} b_{n,d} - \frac{(n+1)(n+2)}{d(2n+d+3)} b_{n+2,d} \tag{4}$$

These recursive relationships suggests that it is possible to express $b_{n,2k+1}, k \geq 1$, as a linear combination of Fourier coefficients $b_{n,1}, b_{n+2,1}, \ldots, b_{n+2k,1}$. Similarly, we can express $b_{n,2k+2}, k \geq 1$, as a linear combination of Legendre coefficients $b_{n,2}, b_{n+2,2}, \ldots, b_{n+2k,2}$.

The aim of this work is to provide closed form expressions of the coefficients appearing in these linear combinations, which was stated as Problem 2 in Gneiting, 2013.

2 Main results

In this section we give the explicit linear combinations for Gegenbauer coefficients in terms of Fourier cosine and Legendre coefficients. The proofs can be found in Sections 4 and 5.

Theorem 2.1. *For integers $k \geq 1$ and $n \geq 0$ the Gegenbauer coefficient $b_{n,2k+1}$ of a function $\psi \in \Psi_{2k+1}$ can be expressed in terms of its Fourier cosine coefficients $b_{n,1}, b_{n+2,1}, \ldots, b_{n+2k,1}$, in that*

$$b_{n,2k+1} = \sum_{i=0}^{k} a_i(n,k) b_{n+2i,1},$$

where the coefficients $a_i(n,k)$ are given by

$$\begin{aligned} a_i(n,k) &= \frac{(-1)^i}{2^k} \binom{k}{i} \frac{(n+k)(n+2i)}{(2k-1)!!} \frac{(n+1)_{(2k-1)}}{(n+i)_{(k+1)}} && \textit{for } (i,n) \neq (0,0) \textit{ and} \\ a_0(0,k) &= 1 && \textit{for } i = n = 0, \end{aligned} \tag{5}$$

where $(2k-1)!! = \prod_{i=1}^{k}(2i-1)$ and $(x)_{(m)} = x(x+1)\cdots(x+m-1)$ denotes the Pochhammer symbol*.*

Example 2.2. Consider $k = 4$. For $n > 0$ we get

$$\begin{aligned} a_0(n,4) &= \kappa(n+4)(n+5)(n+6)(n+7) \\ a_1(n,4) &= -4\kappa(n+2)(n+4)(n+6)(n+7) \\ a_2(n,4) &= 6\kappa(n+1)(n+4)^2(n+7) \\ a_3(n,4) &= -4\kappa(n+1)(n+2)(n+4)(n+6), \\ a_4(n,4) &= \kappa(n+1)(n+2)(n+3)(n+4), \end{aligned}$$

where $\kappa = \frac{1}{1680}$.

One sees that $a_0(n,4)$ and $a_4(n,4)$ can be expressed in simpler forms. In general, for $i=0$ and $i=k$ equation (5) reduces to

$$a_0(n,k) = \frac{1}{2^k(2k-1)!!}(n+k)_{(k)}$$
$$a_k(n,k) = \left(-\frac{1}{2}\right)^k \frac{1}{(2k-1)!!}(n+1)_{(k)}.$$

It might be interesting to note that the value of $\sum_{i=0}^{k} a_i(n,k)$ is either 0 or $\frac{1}{2}$, despite the cumbersome appearance of the explicit formula for $a_i(n,k)$.

Proposition 2.3. For all integers $k \geq 0$ it is

$$\sum_{i=0}^{k} a_i(n,k) = \begin{cases} 0, & n > 0 \\ \frac{1}{2}, & n = 0 \end{cases}.$$

The proof can be found in Section 4.

Now let us turn to the analogous problem of finding a linear combination for $b_{n,2k+2}$, $k \geq 1$, in terms of Legendre coefficients $b_{n,2}, \ldots, b_{n+2k,2}$. This is very similar to the problem above and one suspects similar coefficients.

Theorem 2.4. *For all integers $k \geq 1$ and $n \geq 0$ we have*

$$b_{n,2k+2} = \sum_{i=0}^{k} u_i(n,k) b_{n+2i,2}, \tag{6}$$

where the coefficients $u_i(n,k)$ are given by

$$u_i(n,k) = (-1)^i \frac{(2k-1)!!}{2^k} \binom{k}{i} \binom{2k+n}{n} \frac{1}{(n+i+1/2)_{(k-i)}(n+k+3/2)_{(i)}}. \tag{7}$$

3 Applications

3.1 Continuity

As mentioned in the introduction, several results are known that connect the behaviour of Legendre and Fourier expansion coefficients with the local behaviour of the correlation function. Boas Boas, 1967 analyzed Fourier cosine series with non-negative coefficients, which correspond to members of Ψ_1. If $b_{n,1} \downarrow 0$, he showed that

(i) $\psi \in \Psi_1$ is Hölder continuous with exponent $\gamma \in (0,1)$ if and only if $b_{n,1} = \mathcal{O}(n^{-\gamma-1})$ and

(ii) ψ has fractal index $\gamma \in (0,1)$, meaning that $1 - \psi(\theta) = \mathcal{O}(\theta^\gamma)$ as $\theta \downarrow 0$, if and only if $b_{n,1} = \mathcal{O}(n^{-\gamma-1})$.

These results allow the construction of members of Ψ_{2k+1}, $k \geq 0$, with a given fractal index $\gamma \in (0,1)$. Note that this construction principle works only if the constructed coefficients $b_{n,2k+1}$ are again non-negative, which might be hard to verify.

Example 3.1. Let

$$b_{n,1} = \frac{6}{\pi^2 n^2}, \qquad n \geq 1,$$

and $b_{0,1} = 0$. Obviously, the corresponding function is in Ψ_1^+. For $n, k \geq 1$ a symbolical calculation with Mathematica yields

$$b_{n,2k+1} = \sum_{i=0}^{k} a_i(n,k) b_{n+2i,1} = \frac{3k(n+k)B(n/2,k)^2}{n\pi^2(n+2k)^2 B(n,2k)},$$

where $B(x,y)$ denotes the Beta function. Note that $b_{n,2k+1} > 0$ for all $n, k \geq 1$ showing that the corresponding function is in Ψ_∞^+.

By Stirling's formula for Gamma functions (see 6.1.37 in Abramowitz and Stegun, 1972), we get for fixed y that

$$B(x,y) = \Gamma(y) e^y \left(1 + \frac{y}{x}\right)^{1/2-y-x} x^{-y} \frac{1 + \mathcal{O}(x^{-1})}{1 + \mathcal{O}((x+y)^{-1})}.$$

Since $\left(1 + \frac{y}{x}\right)^{1/2-y-x} \to e^{-y}$ as $x \to \infty$ we see that

$$B(x,y) \approx \Gamma(y) x^{-y},$$

for $x \to \infty$, yielding immediately

$$\frac{B(n/2,k)^2}{B(n,2k)} = \mathcal{O}(1).$$

Consequently, $b_{n,2k+1} = \mathcal{O}(n^{-2})$ for all $k \geq 1$, implying that the higher-dimensional Schoenberg coefficients show the same asymptotic behaviour as $b_{n,1}$.

3.2 Differentiability

Hitczenko and Stein Hitczenko and Stein, 2012 derive for a certain class of stationary and isotropic random fields on the sphere $\mathbb{S}^2$ necessary and sufficient conditions for their corresponding two-dimensional Schoenberg coefficients. In particular, they considered the class of stationary and isotropic random fields on $\mathbb{S}^2$ with correlation function

$$\psi(\theta) = \sum_{n=0}^{\infty} c(n) \frac{2n+1}{2} P_n(\cos\theta), \tag{8}$$

such that the coefficients $c(n)$ fulfill the following: there exist $c_n > 0$, $n \geq 0$, such that for $n > 0$ we have $c(n) = \frac{c_n}{n^{2+\varepsilon}}$, $\varepsilon > 0$, and for $n = 0$ it is $c(0) = c_0$; furthermore c_n is bounded below (above) by $\lambda_1 > 0$ ($\lambda_2 > 0$) and the limit $\lim_{n\to\infty} c_n = c$ exists. They showed that the corresponding isotropic and stationary process is k-times mean square differentiable along any fixed direction if and only if $\varepsilon > 2k$.

Their proof is based on showing that the corresponding isotropic covariance function is $2k$-times differentiable at zero. We can use this result to construct stochastic processes on spheres in even dimensions which have k partial derivatives in any direction. Since we know that $b_{n,2} = c(n)\frac{2n+1}{2}$ we can transfer the assumptions on $c(n)$ readily to $b_{n,2}$ and with this we get a construction principle for higher dimensions.

4 Proof of Theorem 2.1 and Proposition 2.3

Proof of Theorem 2.1: First we consider the case $n > 0$. We proceed by induction for $k \geq 1$.

Let $k = 1$. For all $n \geq 1$ we have

$$b_{n,3} = \frac{1}{2}(n+1)(b_{n,1} - b_{n+2,1}),$$

yielding $a_0(n,1) = \frac{1}{2}(n+1)$ and $a_1(n,1) = -\frac{1}{2}(n+1)$. Inserting $i = 0$ and $k = 1$ into formula (5) immediately yields the same results, proving the claim for $k = 1$.

Suppose we have proven (5) for an arbitrary $k \geq 1$. From this we want to deduce (5) for $k+1$. With (4) we see by comparing coefficients that

$$\begin{aligned}
b_{n,2(k+1)+1} &= \frac{(n+2k)_{(2)}}{2(2k+1)(n+k)} b_{n,2k+1} - \frac{(n+1)_{(2)}}{2(2k+1)(n+k+2)} b_{n+2,2k+1} \\
&= \frac{(n+2k)_{(2)}}{2(2k+1)(n+k)} \sum_{i=0}^{k} a_i(n,k) b_{n+2i,1} - \frac{(n+1)_{(2)}}{2(2k+1)(n+k+2)} \sum_{i=0}^{k} a_i(n+2,k) b_{n+2+2i,1} \\
&= \frac{(n+2k)_{(2)}}{2(2k+1)(n+k)} a_0(n,k) b_{n,1} - \frac{(n+1)_{(2)}}{2(2k+1)(n+k+2)} a_k(n+2,k) b_{n+2(k+1),1} \\
&\quad + \frac{1}{2(2k+1)} \sum_{i=1}^{k} \left[\frac{(n+2k)_{(2)}}{n+k} a_i(n,k) - \frac{(n+1)_{(2)}}{n+k+2} a_{i-1}(n+2,k) \right] b_{n+2i,1}.
\end{aligned}$$

Using the induction hypothesis and the trivial identity

$$(x)_{(k)}(x+k)_{(l)} = (x)_{(k+l)} \qquad \text{for integers } k, l \geq 0, \tag{9}$$

we see that

$$\begin{aligned}
\frac{(n+2k)_{(2)}}{2(2k+1)(n+k)} a_0(n,k) &= \frac{(n+2k)_{(2)}}{2(2k+1)(n+k)} \frac{1}{2^k} \frac{(n+k)n(n+1)_{(2k-1)}}{(2k-1)!!(n)_{(k+1)}} \\
&= \frac{1}{2^{k+1}} \frac{n(n+k+1)(n+1)_{(2k+1)}}{(2k+1)!!(n)_{(k+2)}},
\end{aligned}$$

proving the validity of (5) for $i = 0$ and

$$\begin{aligned}
&-\frac{(n+1)_{(2)}}{2(2k+1)(n+k+2)} a_k(n+2,k) \\
&= -\frac{(n+1)_{(2)}}{2(2k+1)(n+k+2)} \frac{(-1)^k}{2^k} \frac{(n+2+k)(n+2+2k)(n+2+1)_{(2k-1)}}{(2k-1)!!(n+2+k)_{(k+1)}} \\
&= \frac{(-1)^{k+1}}{2^{k+1}} \frac{(n+2+2k)(n+1)_{(2k+1)}}{(2k+1)!!(n+2+k)_{(k+1)}} \\
&= \frac{(-1)^{k+1}}{2^{k+1}} \frac{(n+k+1)(n+2k+2)(n+1)_{(2k+1)}}{(2k+1)!!(n+1+k)_{(k+2)}},
\end{aligned}$$

which shows the validity of (5) for $i = k+1$.

It remains to show for $1 \le i \le k$ that

$$\begin{aligned}
&\frac{(-1)^i}{2^{k+1}}\binom{k+1}{i}\frac{(n+k+1)(n+2i)(n+1)_{(2k+1)}}{(2k+1)!!(n+i)_{(k+2)}}\\
&= \frac{1}{2(2k+1)}\left[\frac{(n+2k)_{(2)}}{n+k}a_i(n,k) - \frac{(n+1)_{(2)}}{n+k+2}a_{i-1}(n+2,k)\right],
\end{aligned} \tag{10}$$

where $a_i(n,k)$ and $a_{i-1}(n+2,k)$ can be expressed as in (5). Plugging the induction hypothesis into (10) we can reformulate this as

$$\begin{aligned}
&\frac{(-1)^i}{2^{k+1}}\binom{k+1}{i}\frac{(n+k+1)(n+2i)(n+1)_{(2k+1)}}{(2k+1)!!(n+i)_{(k+2)}}\\
&= \frac{(-1)^i}{2^{k+1}}\frac{(n+2k)_{(2)}}{n+k}\binom{k}{i}\frac{(n+k)(n+2i)(n+1)_{(2k-1)}}{(2k+1)!!(n+i)_{(k+1)}}\\
&\quad - \frac{(-1)^{i-1}}{2^{k+1}}\frac{(n+1)_{(2)}}{n+k+2}\binom{k}{i-1}\frac{(n+2+k)(n+2i)(n+3)_{(2k-1)}}{(2k+1)!!(n+1+i)_{(k+1)}}.
\end{aligned}$$

By using (9) and canceling factors we see that this is equivalent to

$$\binom{k+1}{i}(n+k+1)\frac{1}{(n+i)_{(k+2)}} = \binom{k}{i}\frac{1}{(n+i)_{(k+1)}} + \binom{k}{i-1}\frac{1}{(n+i+1)_{(k+1)}}.$$

After multiplying with $(n+i)_{(k+2)}$ it remains to show that

$$\binom{k+1}{i}(n+k+1) = \binom{k}{i}(n+i+k+1) + \binom{k}{i-1}(n+i). \tag{11}$$

The right hand side of (11) equals

$$\begin{aligned}
&\frac{k!}{(k-i)!(i-1)!}\left(\frac{n+i+k+1}{i} + \frac{n+i}{k-i+1}\right)\\
&= \frac{k!}{(k+1-i)!i!}\big[(k-i+1)(n+i+k+1) + i(n+i)\big]\\
&= \frac{k!}{(k+1-i)!i!}(k+1)(n+k+1) = \binom{k+1}{i}(n+k+1),
\end{aligned}$$

showing the validity of (11) and we are done.

Now consider the case $n = 0$. As (3) and (4) show, the coefficient $b_{n,d+2}$ satisfies for all $n \ge 0$ and all $d \ge 2$ the same recursive property. This shows that $a_i(0,k)$ is given by (5) for all $i > 0$. If $n = 0$ we can reformulate (4) and together with (3) we see that the following holds for all $d \ge 1$

$$b_{0,d+2} = b_{0,d} - \frac{2}{d(d+3)}b_{2,d}.$$

Using this relationship recursively, we see that for all $d \ge 1$ it holds that

$$b_{0,d+2} = b_{0,1} - R,$$

where the remainder term R does not depend on $b_{0,1}$. This shows that $a_0(0,k) = 1$ for all $k \ge 1$. $\square$

Proof of Proposition 2.3: Let $n > 0$. Using $(n+i)_{(k+1)} = \binom{n+i+k}{k+1}(k+1)!$ and Theorem 2.1 we see that

$$\begin{aligned}
\sum_{i=0}^{k} a_i(n,k) &= \frac{(n+k)(n+1)_{(2k-1)}}{2^k(2k-1)!!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{n+2i}{(n+i)_{(k+1)}} \\
&= \frac{(n+k)(n+1)_{(2k-1)}}{2^k(k+1)!(2k-1)!!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{n+2i}{\binom{n+i+k}{k+1}}.
\end{aligned}$$

Hence, it suffices to prove that

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{n+2i}{\binom{n+i+k}{k+1}} = 0,$$

which is equivalent to

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{n}{\binom{n+i+k}{k+1}} = -2\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{i}{\binom{n+i+k}{k+1}}. \tag{12}$$

Now the left-hand side of (12) equals

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{n}{\binom{n+i+k}{k+1}} = n\frac{k+1}{2k+1}\frac{1}{\binom{2k+n}{n-1}} = \frac{k+1}{\binom{2k+n}{n}},$$

where the first equality is due to the following result of R. Frisch which can be found, for example, as Note 21 in Netto, 1927

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{1}{\binom{b+i}{c}} = \frac{c}{k+c}\frac{1}{\binom{k+b}{b-c}}, \tag{13}$$

where $b \geq c$ are positive integers.

For the right-hand side of (12) we get in a very similar way

$$\begin{aligned}
-2\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{i}{\binom{n+i+k}{k+1}} &= -2\sum_{i=0}^{k}(-1)^i\frac{k!}{(k-i)!(i-1)!}\frac{1}{\binom{n+i+k}{k+1}} \\
&= -2k\sum_{i=0}^{k}(-1)^i\binom{k-1}{i-1}\frac{1}{\binom{n+i+k}{k+1}} \\
&= -2k\sum_{i=1}^{k}(-1)^i\binom{k-1}{i-1}\frac{1}{\binom{n+i+k}{k+1}} \\
&= 2k\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}\frac{1}{\binom{n+i+1+k}{k+1}} \\
&= 2k\frac{k+1}{2k}\frac{1}{\binom{2k+n}{n}} = \frac{k+1}{\binom{2k+n}{n}},
\end{aligned}$$

where we use (13) for the final equality, thereby showing (12).

Now consider the case $n=0$. For $i>0$, equation (5) simplifies to

$$a_i(0,k) = (-1)^i \binom{k}{i} \frac{1}{\binom{k+i}{k}},$$

which is also valid for $i=0$, since in this case it reduces to 1. Hence, by using (13) with $b=c=k$ we get

$$\sum_{i=0}^{k} a_i(0,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{1}{\binom{k+i}{k}} = \frac{1}{2}.$$

□

5 Proof of Theorem 2.4

Proof of Theorem 2.4: We proceed by induction. Let $k=1$. Then

$$b_{n,4} = \frac{1}{2}(n+1)_{(2)} \left[\frac{1}{2n+1} b_{n,2} - \frac{1}{2n+5} b_{n+2,2}\right],$$

implying $u_0(n,1) = \frac{1}{2(2n+1)}(n+1)_{(2)}$ and $u_1(n,1) = -\frac{1}{2(2n+5)}(n+1)_{(2)}$. Inserting $k=1$, and $i=0$ and $i=1$ respectively in equation (7) proves the claim for $k=1$.

Suppose we have proven (7) for a $k \geq 1$. We use this to show the validity of (7) for $k+1$. Using (3) and (4) and the induction hypothesis we see that

$$\begin{aligned}
b_{n,2(k+1)+2} &= \frac{(n+2k+1)_{(2)}}{2(k+1)(2n+2k+1)} b_{n,2k+2} - \frac{(n+1)_{(2)}}{2(k+1)(2n+2k+5)} b_{n+2,2k+2} \\
&= \frac{(n+2k+1)_{(2)}}{2(k+1)(2n+2k+1)} \sum_{i=0}^{k} u_i(n,k) b_{n+2i,2} - \frac{(n+1)_{(2)}}{2(k+1)(2n+2k+5)} \sum_{i=0}^{k} u_i(n+2,k) b_{n+2+2i,2} \\
&= \frac{(n+2k+1)_{(2)}}{2(k+1)(2n+2k+1)} u_0(n,k) b_{n,2} - \frac{(n+1)_{(2)}}{2(k+1)(2n+2k+5)} u_k(n+2,k) b_{n+2(k+1),2} \\
&\quad + \sum_{i=1}^{k} \left[\frac{(n+2k+1)_{(2)}}{2(k+1)(2n+2k+1)} u_i(n,k) - \frac{(n+1)_{(2)}}{2(k+1)(2n+2k+5)} u_{i-1}(n+2,k)\right] b_{n+2i,2}.
\end{aligned}$$

Using $(2k-1)!! = \frac{(2k)!}{2^k k!}$ and (9) we see that

$$\begin{aligned}
\frac{(n+2k+1)_{(2)}}{2(k+1)(2n+2k+1)} u_0(n,k) &= \frac{(n+2k+1)_{(2)}(2k-1)!!}{2^{k+1}(k+1)(2n+2k+1)} \binom{2k+n}{n} \frac{1}{(n+1/2)_{(k)}} \\
&= \frac{1}{2^{k+1}2^{k+1}} \frac{(2k+2+n)!}{(k+1)!n!} \frac{1}{(n+1/2)_{(k+1)}} \\
&= \frac{(2k+1)!!}{2^{k+1}} \binom{2k+2+n}{n} \frac{1}{(n+1/2)_{(k+1)}},
\end{aligned}$$

proving the claim for $i=0$. In a very similar way we get

$$
\begin{aligned}
&-\frac{(n+1)_{(2)}}{2(k+1)(2n+2k+5)}u_k(n+2,k)\\
&=(-1)^{k+1}\frac{(n+1)_{(2)}(2k-1)!!}{2^{k+1}(k+1)(2n+2k+5)}\binom{2k+n+2}{n+2}\frac{1}{(n+k+7/2)_{(k)}}\\
&=(-1)^{k+1}\frac{1}{2^{k+1}2^{k+1}}\frac{(2k+2+n)!}{(k+1)!n!}\frac{1}{(n+k+5/2)_{(k/2)}}\\
&=(-1)^{k+1}\frac{(2k+1)!!}{2^{k+1}}\binom{2k+2+n}{n}\frac{1}{(n+k+5/2)_{(k/2)}},
\end{aligned}
$$

confirming the claim for $i=k+1$.

Now let $1\le i\le k$. We have to show that

$$
\begin{aligned}
u_i(n,k+1)&=(-1)^i\frac{(2k+1)!!}{2^{k+1}}\binom{k+1}{i}\binom{2k+2+n}{n}\frac{1}{(n+i+1/2)_{(k+1-i)}(n+k+1+3/2)_{(i)}}\\
&=\frac{(n+2k+1)_{(2)}}{2(k+1)(2n+2k+1)}u_i(n,k)-\frac{(n+1)_{(2)}}{2(k+1)(2n+2k+5)}u_{i-1}(n+2,k),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&(-1)^i\frac{(2k+1)!!}{2^{k+1}}\binom{k+1}{i}\binom{2k+2+n}{n}\frac{1}{(n+i+1/2)_{(k+1-i)}(n+k+1+3/2)_{(i)}}\\
&=\frac{(-1)^i(n+2k+1)_{(2)}(2k-1)!!}{2^{k+1}(k+1)(2n+2k+1)}\binom{k}{i}\binom{2k+n}{n}\frac{1}{(n+i+1/2)_{(k-i)}(n+k+3/2)_{(i)}}\\
&-\frac{(-1)^{i-1}(n+1)_{(2)}(2k-1)!!}{2^{k+1}(k+1)(2n+2k+5)}\binom{k}{i-1}\binom{2k+n+2}{n+2}\frac{1}{(n+i+1+1/2)_{(k-i+1)}(n+2+k+3/2)_{(i-1)}}.
\end{aligned}
\tag{14}
$$

Using (9) and doing similar manipulations as in the cases $i=0$ and $i=k+1$ we see that (14) is equivalent to

$$
\begin{aligned}
&\binom{k+1}{i}\frac{1}{(n+i+1/2)_{(k+1-i)}(n+k+5/2)_{(i)}}\\
&=\binom{k}{i}\frac{1}{(n+i+1/2)_{(k+1-i)}(n+k+3/2)_{(i)}}+\binom{k}{i-1}\frac{1}{(n+i+3/2)_{(k+1-i)}(n+k+5/2)_{(i)}}.
\end{aligned}
$$

Multiplying with $(n+i+1/2)_{(k+1-i)}(n+k+5/2)_{(i)}$ illustrates that we have to show

$$
\binom{k+1}{i}=\binom{k}{i}\frac{n+k+i+3/2}{n+k+3/2}+\binom{k}{i-1}\frac{n+i+1/2}{n+k+3/2}.
$$

Simplifying the right hand side yields

$$
\begin{aligned}
&\binom{k}{i}\frac{(n+k+i+3/2)(k-i+1)+i(n+i+1/2)}{(n+k+3/2)(k-i+1)}\\
&\qquad=\binom{k}{i}\frac{(k+1)(n+k+3/2)}{(n+k+3/2)(k-i+1)}=\binom{k+1}{i},
\end{aligned}
$$

and we are done. $\square$

Acknowledgements

I would like to thank Tilmann Gneiting for helpful discussions and comments.

References

Abramowitz, M. and I. A. Stegun (1972). *Handbook of Mathematical Functions*. Dover.

Banerjee, S. (June 2005). "On geodetic distance computations in spatial modeling". In: *Biometrics* 61.2, pp. 617–25. ISSN: 0006-341X. DOI: 10.1111/j.1541-0420.2005.00320.x. URL: http://www.ncbi.nlm.nih.gov/pubmed/16011712.

Boas, R. P. (1967). "Fourier series with positive coefficients". In: *Journal of Mathematical Analysis and Applications* 17, pp. 463–483.

Cavoretto, R. and A. De Rossi (July 2010). "Fast and accurate interpolation of large scattered data sets on the sphere". In: *Journal of Computational and Applied Mathematics* 234, pp. 1505–1521. ISSN: 03770427. DOI: 10.1016/j.cam.2010.02.031. URL: http://linkinghub.elsevier.com/retrieve/pii/S0377042710001299.

Chen, D., V. A. Menegatto, and X. Sun (2003). "A necessary and sufficient condition for strictly positive definite functions on spheres". In: *Proceedings of the American Mathematical Society* 131.9, pp. 2733–2740.

Daley, D. J. and E. Porcu (2013). "Dimension walks and schoenberg spectral measures". In: *Proceedings of the American Mathematical Society* in press.

Fasshauer, G. E. and L. L. Schumaker (1998). "Scattered Data Fitting on the Sphere". In: *Mathematical Methods for Curves and Surfaces: Volume II*. Ed. by M. Dæhlen, T. Lyche, and L. L. Schumaker. Nashville: Vanderbilt University Press, pp. 117–166.

Gneiting, T. (2013). "Strictly and non-strictly positive definite functions on spheres". In: *Bernoulli* in press. arXiv:arXiv:1111.7077v4.

Hansen, L. V., T. L. Thorarinsdottir, and T. Gneiting (2011). "Lévy particles: Modelling and simulating star-shaped random sets". In: *CSGB Research Report*.

Hitczenko, M. and M. L. Stein (Jan. 2012). "Some theory for anisotropic processes on the sphere". In: *Statistical Methodology* 9, pp. 211–227. ISSN: 15723127. DOI: 10.1016/j.stamet.2011.01.01 URL: http://linkinghub.elsevier.com/retrieve/pii/S157231271100013X.

Huang, C., H. Zhang, and S. Robeson (2011). "On the validity of covariance and variogram functions on the sphere". In: *Mathematical Geosciences* 43, pp. 721–733.

Jones, R. H. (1963). "Stochastic processes on a sphere". In: *Annals of Mathematical Statistics* 34, pp. 213–218.

Mathéron, G. (1965). *Les variables régionalisées et leur estimation*. Paris: Masson.

Netto, E. (1927). *Lehrbuch der Combinatorik*. Second. New York: Chelsea Publishing Company.

Schoenberg, I. J. (1942). "Positive definite functions on spheres". In: *Duke Mathematical Journal* 9, pp. 96–108.

Wendland, H. (Dec. 1995). "Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree". In: *Advances in Computational Mathematics* 4.1, pp. 389–396. ISSN: 1019-7168. DOI: 10.1007/BF02123482. URL: http://www.springerlink.com/ind

Xu, Y. and E. W. Cheney (1992). "Strictly Positive Definite Functions on Spheres". In: *Proceedings of the American Mathematical Society* 116, pp. 977–981.

Ziegel, J. (2013). "Convolution roots and differentiability of isotropic positive definite functions on spheres". In: *Proceedings of the American Mathematical Society* in press. arXiv:arXiv:1201.5833v2.